

# Probabilistic interpretation of a system of quasilinear elliptic partial differential equations under Neumann boundary conditions

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A probabilistic interpretation of a system of second order quasilinear elliptic partial differential equations under a Neumann boundary condition is obtained by introducing a kind of backward stochastic differential equations in the infinite horizon case. In the same time, a simple proof for the existence and uniqueness result of the classical solution of that Neumann problem is given.

reflecting Brownian motion \* local time \* backward stochastic differential equation \* Neumann boundary condition

## 1. Introduction

It is well known that a solution of a linear second order elliptic equation with a Neumann boundary condition can be formulated in terms of reflecting Brownian motion and its boundary local time (see e.g. Freidlin, 1985; or Hsu, 1985). Can we obtain a similar interpretation of a system of quasilinear elliptic equations with a Neumann boundary condition? This is the aim of our paper.

More precisely, we consider the following linear Neumann boundary value problem:

$$\begin{aligned} (\tfrac{1}{2}\Delta - \mu)u &= 0, & x \in Q, \\ \partial u / \partial n &= \varphi, & x \in \partial Q = G, \end{aligned} \quad (1.1)$$

where  $Q$  is a bounded domain in  $\mathbb{R}^n$  with a  $C^3$  boundary  $G$ ,  $\mu > 0$  is a constant. Then we know that if  $\varphi \in C^2(\bar{G})$ , then a  $C^2(\bar{Q})$  solution of (1.1) can be given a probabilistic interpretation (Freidlin, 1985; Hsu, 1985),

$$u(x) = \tfrac{1}{2} E^x \int_0^{+\infty} e^{-\mu t} \varphi(y(t)) L(dt) \quad (1.2)$$

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where  $y(\cdot)$  is the standard reflecting Brownian motion on  $\bar{Q}$  with the boundary local time  $L(\cdot)$  (see Section 2 for more details).

The  $C^2(\bar{Q})$  solution of (1.1) can be interpreted with a different point of view. Consider the following one-dimensional linear backward SDE in the infinite horizon case (the existence and uniqueness for  $(p(\cdot), q(\cdot))$  will be proved in Section 1):

$$\begin{aligned} dp(t) &= -\frac{1}{2}\varphi(y(t))L(dt) + \mu p(t)dt + q(t)dB(t), \\ (L^2) \lim_{t \rightarrow +\infty} e^{-\mu t} p(t) &= 0, \end{aligned} \quad (1.3)$$

where  $B(\cdot)$  is a  $\mathbb{R}^n$ -Brownian motion whose relation with  $y(\cdot)$  is described by the generalized Tanaka formula,

$$\begin{aligned} dy(t) &= dB(t) - \frac{1}{2}n(y(t))L(dt), \\ y(0) &= x. \end{aligned} \quad (1.4)$$

Then we will see in Section 2 that

$$u(x) = E^x p(0) = p(0) \quad (1.5)$$

and this interpretation (1.5) is equivalent to the classical one. But this new interpretation can be applied to a more general case.

Instead of linear backward SDE (1.3), we shall consider the following nonlinear backward SDE: to find an adapted pair  $(p(\cdot), q(\cdot))$  with values in  $(\mathbb{R}^m, \mathbb{R}^{m \times n})$ , which is a unique solution of

$$\begin{aligned} dp(t) &= -\frac{1}{2}\varphi(y(t))L(dt) - f(y(t), p(t))dt + q(t)dB(t), \\ (L^2) \lim_{t \rightarrow +\infty} e^{-\mu t} p(t) &= 0, \end{aligned} \quad (1.6)$$

where  $f$  is a given function defined on  $\mathbb{R}^n \times \mathbb{R}^m$  with values in  $\mathbb{R}^m$ ,  $\varphi$  is a given function defined on  $G$  with values in  $\mathbb{R}^m$ . Then we shall show that a  $C^2(\bar{Q}; \mathbb{R}^m)$  solution of the system of quasilinear elliptic PDEs with a Neumann boundary condition

$$\begin{aligned} \frac{1}{2}\Delta u + f(x, u) &= 0, & x \in Q, \\ \partial u / \partial n &= \varphi, & x \in G, \end{aligned} \quad (1.7)$$

can be given a probabilistic interpretation,

$$u(x) = E^x p(0) = p(0) \quad (1.8)$$

where  $(p(\cdot), q(\cdot))$  is the unique solution of (1.6).

The nonlinear backward SDE in the finite horizon case was studied by Pardoux and Peng (1990), Hu and Peng (1991).

The idea of interpreting the solution of systems of quasilinear PDEs by introducing backward SDEs is due to Peng (1991), where a probabilistic interpretation of a system of quasilinear parabolic PDEs is given. It is interesting to indicate that a

simplified version of (1.6) is studied in Duffie and Epstein (1989) and applications were found in the mathematical finance.

The paper is organized as follows. In Section 2, we recall some facts and discuss the linear case, refine the formula (1.2) in a simple way. In Section 3, we treat the problem in a simplified case when  $\varphi = 0$ . In Section 4, we treat the general one. And in the last section, we conclude our paper with an existence and uniqueness result for (1.7) by use of a Leray-Schauder type result in Ural'ceva (1962).

## 2. Preliminaries and linear case

In this section, we first recall some facts on the reflecting Brownian motion and its local time.

Let  $Q$  be a bounded domain in  $\mathbb{R}^n$  with a  $C^3$ -boundary  $G$ . The standard reflecting Brownian motion  $y = \{y(t), t \geq 0\}$  on  $G$  is a diffusion process with state space  $\bar{Q}$  whose transition density function satisfies the parabolic PDE

$$\begin{aligned} \frac{\partial}{\partial t} p(t, x, y) &= \frac{1}{2} \Delta_x p(t, x, y), \quad (t, x, y) \in (0, +\infty) \times Q \times Q, \\ \lim_{t \downarrow 0} p(t, x, y) &= \delta_y(x), \quad (x, y) \in \bar{Q} \times \bar{Q}, \\ \frac{\partial}{\partial n_x} p(t, x, y) &= 0, \quad (t, x, y) \in (0, +\infty) \times G \times \bar{Q}, \end{aligned} \quad (2.1)$$

where  $\Delta_x$  is the Laplacian on  $x$  variables,  $n_x$  is the outward unit normal vector at  $x \in \partial D$ , and  $\delta_y$  is the Dirac delta function at  $y$ .

The boundary local time of the reflecting Brownian motion is defined by

$$L(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^t I_{D_\varepsilon(y(s))} ds \quad (2.2)$$

where

$$D_\varepsilon = \{x \in \bar{Q} : d(x, \partial Q) \leq \varepsilon\}. \quad (2.3)$$

The limit in (2.2) exists in  $L^2$  as well as a.s. The boundary local time  $L(\cdot)$  has the following properties which will be useful to us:

$$\sup_{x \in \bar{Q}} E^x |L(t)|^2 \leq Kt, \quad K > 0 \text{ is a constant.} \quad (2.4)$$

$$p^x \{L(t) > 0 \text{ for all } t > 0\} = 0 \text{ or } 1 \quad (2.5)$$

depending on whether  $x \in Q$  or  $x \in \partial Q$ .

$L(\cdot)$  is a continuous additive functional of  $y(\cdot)$ .

$$\text{With probability 1, it increases only when } y(t) \in \partial D. \quad (2.6)$$

The generalized Tanaka formula holds:

$$\begin{aligned} dy(t) &= dB(t) - \frac{1}{2}n(y(t))L(dt), \\ y(0) &= x, \end{aligned} \quad (2.7)$$

where  $B(\cdot)$  is a standard Brownian motion on  $\mathbb{R}^n$ .

For proofs of the results stated above, the reader is referred to Hsu (1984) or Sato and Ueno (1965).

Now we turn to study the eq. (1.3). We denote

$$\mathcal{F}_t = \sigma(B(s), s \leq t).$$

For  $T \in (0, +\infty]$ , we denote by  $M^2(0, T; \mathbb{R}^m)$  (resp.  $M^2(0, T; \mathbb{R}^{m \times n})$ ) the set of  $\mathbb{R}^m$ -valued (resp.  $\mathbb{R}^{m \times n}$ -valued)  $\mathcal{F}_t$ -progressively measurable processes  $v(\cdot)$  such that

$$E \int_0^T |v(s)|^2 ds < +\infty,$$

and by  $M^{2,\mu}(0, +\infty; \mathbb{R}^{m \times n})$  the set of the processes  $v(\cdot)$  which have the property that the process  $\{v(s)e^{-\mu s}, s \geq 0\}$  is in  $M^2(0, +\infty; \mathbb{R}^{m \times n})$ .

**Definition 2.1.** We say that a pair  $(p(\cdot), q(\cdot))$  is a solution of (1.6) if

- (1)  $p(\cdot)$  is a  $\mathcal{F}_t$ -progressively measurable process with values in  $\mathbb{R}^m$ ;
- (2)  $q(\cdot) \in M^{2,\mu}(0, +\infty; \mathbb{R}^{m \times n})$ ;
- (3)  $(p(\cdot), q(\cdot))$  satisfies (1.6).

**Theorem 2.2.** Assume  $\varphi \in C^2(G; \mathbb{R}^m)$ . Then (1.3) has a unique solution  $(p(\cdot), q(\cdot))$ .

**Proof.** (1) *Uniqueness.* If we have two solutions  $(p_1(\cdot), q_1(\cdot)), (p_2(\cdot), q_2(\cdot))$  of (1.3), then we set

$$\tilde{p}(\cdot) = p_1(\cdot) - p_2(\cdot), \quad \tilde{q}(\cdot) = q_1(\cdot) - q_2(\cdot),$$

we would have

$$d\tilde{p}(t) = \mu\tilde{p}(t) dt + \tilde{q}(t) dB(t),$$

$$(L^2) \lim_{t \rightarrow +\infty} e^{-\mu t} p(t) = 0.$$

From the variation-of-constants formula,

$$e^{-\mu t} \tilde{p}(t) = \tilde{p}(0) + \int_0^t e^{-\mu s} \tilde{q}(s) dB(s),$$

$$E|e^{-\mu t} \tilde{p}(t)|^2 = E|\tilde{p}(0)|^2 + E^x \int_0^t |e^{-\mu s} \tilde{q}(s)|^2 ds.$$

From which we can easily obtain the uniqueness.

(2) *Existence.* Set

$$p_1(t) = \frac{1}{2} E \left( \int_t^{+\infty} e^{-\mu s} \varphi(y(s)) L(ds) \mid \mathcal{F}_t \right)$$

then

$$p_1(t) = E \left( \frac{1}{2} \int_0^{+\infty} e^{-\mu s} \varphi(y(s)) L(ds) \mid \mathcal{F}_t \right) - \frac{1}{2} \int_0^t e^{-\mu s} \varphi(y(s)) L(ds).$$

From the martingale representation theorem (cf. Ikeda and Watanabe, 1981) there exists a  $q_1(\cdot) \in M^2(0, +\infty; \mathbb{R}^{m \times n})$  such that

$$\begin{aligned} & E \left( \frac{1}{2} \int_0^{+\infty} e^{-\mu s} \varphi(y(s)) L(ds) \mid \mathcal{F}_t \right) \\ &= \frac{1}{2} E \int_0^{+\infty} e^{-\mu s} \varphi(y(s)) L(ds) + \int_0^t q_1(s) dB(s), \end{aligned} \quad (2.8)$$

so

$$\begin{aligned} dp_1(t) &= -\frac{1}{2} e^{-\mu t} \varphi(y(t)) L(dt) + q_1(t) dB(t), \\ (L^2) \lim_{t \rightarrow +\infty} p_1(t) &= 0. \end{aligned}$$

Now set

$$p(t) = e^{\mu t} p_1(t), \quad q(t) = e^{\mu t} q_1(t).$$

It is easy to verify that  $(p(\cdot), q(\cdot))$  is a solution of (1.3).  $\square$

**Theorem 2.3.** Assume  $\varphi \in C^2(G; \mathbb{R}^m)$ . Then, the  $C^2(\bar{Q}; \mathbb{R}^m)$  solution of the linear system of PDEs (1.1) has the following interpretation:

$$u(x) = E^x p(0) = p(0) \quad (2.9)$$

where  $(p(\cdot), q(\cdot))$  is the unique solution of (1.3).

**Proof.** Because  $\varphi \in C^2(G; \mathbb{R}^m)$ , there exists a  $C^2(\bar{Q}; \mathbb{R}^m)$  solution  $u(\cdot)$  to (1.1). We apply Itô's formula to  $u(y(t))$  and utilize the generalized Tanaka formula, we obtain that

$$\begin{aligned} du(y(t)) &= \frac{1}{2} \Delta u(y(t)) dt + \nabla u(y(t)) dy(t) \\ &= \mu u(y(t)) dt - \frac{1}{2} \nabla u(y(t)) \cdot n(y(t)) L(dt) + \nabla u(y(t)) dB(t) \\ &= -\frac{1}{2} \frac{\partial u}{\partial n}(y(t)) L(dt) + \mu u(y(t)) dt + \nabla u(y(t)) dB(t) \\ &= -\frac{1}{2} \varphi(y(t)) L(dt) + \mu u(y(t)) dt + \nabla u(y(t)) dB(t). \end{aligned}$$

Clearly  $(u(y(\cdot)), \nabla u(y(\cdot)))$  is a solution of (1.3), we get

$$p(t) = u(y(t)).$$

Then

$$p(0) = u(y(0)) = u(x). \quad \square$$

**Corollary 2.4.** Assume  $\varphi \in C^2(G; \mathbb{R}^m)$ . Then the  $C^2(\bar{Q}; \mathbb{R}^m)$  solution  $u(\cdot)$  of (1.1) can also be given by

$$u(x) = \frac{1}{2} E^x \int_0^{+\infty} e^{-\mu t} \varphi(y(t)) L(dt). \quad (2.10)$$

**Proof.** Let  $(p(\cdot), q(\cdot))$  be the unique solution of (1.3), then by the variation of constants formula,

$$e^{-\mu t} p(t) = p(0) - \frac{1}{2} \int_0^t e^{-\mu s} \varphi(y(s)) L(ds) + \int_0^t e^{-\mu s} q(s) dB(s),$$

so

$$p(0) = E^x p(0) = E^x e^{-\mu t} p(t) + \frac{1}{2} E^x \int_0^t e^{-\mu s} \varphi(y(s)) L(ds),$$

from which we can obtain (2.10) by letting  $t \rightarrow +\infty$  and (2.9).  $\square$

**Remark 2.5.** In fact,  $y(\cdot)$ ,  $B(\cdot)$ ,  $L(\cdot)$ ,  $p(\cdot)$ ,  $q(\cdot)$ ,  $\{\mathcal{F}_t, t \geq 0\}$  are all dependent of  $x \in \bar{Q}$ , but for simplicity of the exposition, we omit the variable  $x$  in notations.

### 3. Simplified case: $\varphi = 0$

In this section, we begin to study (1.6) in a simplified case when  $\varphi = 0$ . We assume that:

$f$  is a continuous function defined on  $\mathbb{R}^n \times \mathbb{R}^m$  with values in  $\mathbb{R}^m$ . (3.11)

There exists constants  $c > 0$ ,  $\mu > 0$  such that

$$|f(x, v_1) - f(x, v_2)| \leq c|v_1 - v_2|, \quad (3.2)$$

$$\langle f(x, v_1) - f(x, v_2), v_1 - v_2 \rangle \leq -\mu|v_1 - v_2|^2,$$

$\forall x \in \mathbb{R}^n, \forall v_1, v_2 \in \mathbb{R}^m$ .

Now we can state the main result of this section.

**Theorem 3.1.** Assume (3.1), (3.2). Then there exists a unique solution  $(p(\cdot), q(\cdot))$  of

$$\begin{aligned} dp(t) &= -f(y(t), p(t)) dt + q(t) dB(t), \\ (L^2) \lim_{t \rightarrow +\infty} e^{-\mu t} p(t) &= 0. \end{aligned} \quad (3.3)$$

Furthermore, we have

$$\sup_{x \in \bar{Q}} E^x |p(t)|^2 \leq M^2 / \mu^2 \quad (3.4)$$

where

$$M = \max_{x \in \bar{Q}} |f(x, 0)|.$$

For the proof of this theorem, we must first establish a Gronwall type inequality.

**Lemma 3.2.** Assume that  $a(\cdot) \in C([0, T]; \mathbb{R}^+)$ ,  $b(\cdot) \in C([0, T]; \mathbb{R}^+)$ ,

$$a^2(t) \leq 2 \int_t^T a(s)b(s) ds \quad \forall t \in [0, T]. \quad (3.5)$$

Then we would have

$$a(t) \leq \int_t^T b(s) ds \quad \forall t \in [0, T]. \quad (3.6)$$

**Proof.** Set

$$v(t) = 2 \int_t^T a(s)b(s) ds, \quad t \in [0, T].$$

Then

$$\begin{aligned} a^2(t) &\leq v(t), & a(t) &\leq \sqrt{v(t)}, \\ v(T) &= 0, \\ dv(t)/dt &= -2a(t)b(t) \geq -2\sqrt{v(t)}b(t), \\ \int_t^T \frac{dv(s)}{2\sqrt{v(s)}} &\geq - \int_t^T b(s) ds. \end{aligned}$$

So we obtain that

$$-\sqrt{v(t)} \geq - \int_t^T b(s) ds$$

which implies (3.6).  $\square$

Now we prove Theorem 3.1.

**Proof of Theorem 3.1.** (1) *Existence.* For  $k = 1, 2, \dots$ , we first apply the existence and uniqueness result in the finite horizon case in Pardoux and Peng (1990) or Hu and Peng (1991), we know that there exists a unique pair  $(p_k(\cdot), q_k(\cdot)) \in M^2(0, k; \mathbb{R}^m) \times M^2(0, k; \mathbb{R}^{m \times n})$  such that

$$\begin{aligned} dp_k(t) &= -f(y(t), p_k(t)) dt + q_k(t) dB(t), \quad t \in [0, k], \\ p_k(k) &= 0. \end{aligned} \quad (3.7)$$

Applying the Itô's formula to  $|e^{-\mu t} p_k(t)|^2$ , we obtain that

$$\begin{aligned} d(|e^{-\mu t} p_k(t)|^2) &= 2(e^{-\mu t})^2 [(-\mu |p_k(t)|^2 - \langle p_k(t), f(y(t), p_k(t)) \rangle) dt \\ &\quad + \langle p_k(t), q_k(t) \rangle dB(t)] + |e^{-\mu t} q_k(t)|^2 dt. \end{aligned}$$

Integrating it and taking expectations, we have

$$\begin{aligned} E^x |e^{-\mu t} p_k(t)|^2 + E^x \int_t^k |e^{-\mu s} q_k(s)|^2 ds \\ = 2E^x \int_t^k (e^{-\mu s})^2 [\mu |p_k(s)|^2 + \langle f(y(s), p_k(s)), p_k(s) \rangle] ds. \end{aligned}$$

From assumption (3.2), we have

$$\begin{aligned} E^x |e^{-\mu t} p_k(t)|^2 + E^x \int_t^k |e^{-\mu s} q_k(s)|^2 ds \\ \leq 2E^x \int_t^k \langle e^{-\mu s} f(y(s), 0), e^{-\mu s} p_k(s) \rangle ds \\ \leq 2 \int_t^k [E^x |e^{-\mu s} p_k(s)|^2]^{1/2} [E^x |e^{-\mu s} f(y(s), 0)|^2]^{1/2} ds. \end{aligned}$$

So from Lemma 3.2, we obtain that

$$[E^x |e^{-\mu t} p_k(t)|^2]^{1/2} \leq \int_t^k [E^x |e^{-\mu s} f(y(s), 0)|^2]^{1/2} ds \leq \frac{M}{\mu} e^{-\mu t}.$$

We conclude that

$$E^x |e^{-\mu t} p_k(t)|^2 \leq \frac{M^2}{\mu^2} e^{-2\mu t}$$

and

$$E^x \int_t^k |e^{-\mu s} q_k(s)|^2 ds \leq \frac{M^2}{\mu^2} e^{-2\mu t}.$$

Now we prove that for  $t$  fixed,  $k \geq l \geq t$ ,

$$E^x |e^{-\mu t} (p_k(t) - p_l(t))|^2 + E^x \int_t^{+\infty} |e^{-\mu s} (q_k(s) - q_l(s))|^2 ds \rightarrow 0 \quad (3.8)$$

( $k \rightarrow +\infty, l \rightarrow +\infty$ ) where we denote that for  $t > k$ ,  $p_k(t) = 0$ ,  $q_k(t) = 0$  and for  $t > l$ ,  $p_l(t) = 0$ ,  $q_l(t) = 0$ . Set

$$\tilde{f}(t) = \begin{cases} f(y(t), p_l(t)), & 0 \leq t \leq l, \\ 0, & t > l. \end{cases}$$

Then

$$\begin{aligned} d(p_k - p_l)(t) &= -(f(y(t), p_k(t)) - \tilde{f}(t))dt + (q_k(t) - q_l(t)), \quad t \in [0, k], \\ (p_k - p_l)(k) &= 0. \end{aligned}$$



So using the same method as before, we obtain that for  $t \leq l \leq k$ ,

$$\begin{aligned} & E^x |e^{-\mu t} (p_k - p_l)(t)|^2 + E^x \int_t^k |e^{-\mu s} (q_k(s) - q_l(s))|^2 ds \\ &= 2E^x \int_t^k (e^{-\mu s})^2 [\mu |p_k(s) - p_l(s)|^2 \\ &\quad + \langle f(y(s), p_k(s)) - \tilde{f}(s), p_k(s) - p_l(s) \rangle] ds, \end{aligned}$$

from which we obtain that, for  $t \leq l \leq k$ ,

$$\begin{aligned} & E^x |e^{-\mu t} (p_k(t) - p_l(t))|^2 + E^x \int_t^{+\infty} |e^{-\mu s} (q_k(s) - q_l(s))|^2 ds \\ &= 2E^x \int_t^l (e^{-\mu s})^2 [\mu |p_k(s) - p_l(s)|^2 \\ &\quad + \langle f(y(s), p_k(s)) - f(y(s), p_l(s)), p_k(s) - p_l(s) \rangle] ds \\ &\quad + 2E^x \int_l^k (e^{-\mu s})^2 [\mu |p_k(s)|^2 + \langle f(y(s), p_k(s)), p_k(s) \rangle] ds \\ &\leq 2E^x \int_t^k (e^{-\mu s})^2 \langle p_k(s), f(y(s), 0) \rangle ds \\ &\leq 2 \int_t^k [E^x |e^{-\mu s} p_k(s)|^2]^{1/2} [E^x |e^{-\mu s} f(y(s), 0)|^2]^{1/2} ds \\ &\leq 2 \int_t^k \frac{M}{\mu} e^{-\mu s} M e^{-\mu s} ds = \frac{2M^2}{\mu} \int_t^k e^{-2\mu s} ds. \end{aligned}$$

Now (3.8) is proved.

From (3.8), we know that if  $t$  fixed, then  $e^{-\mu t} p_k(t)$  is a Cauchy sequence in  $L^2(\Omega; \mathbb{R}^m)$  and  $q_k(\cdot)$  is a Cauchy sequence in  $M^{2,\mu}(0, +\infty; \mathbb{R}^{m \times n})$ , so there exists  $p(\cdot)$  which is  $\mathcal{F}_t$ -progressively measurable and  $q(\cdot) \in M^{2,\mu}(0, +\infty; \mathbb{R}^{m \times n})$ , such that

$$E |e^{-\mu t} (p_k(t) - p(t))|^2 + E^x \int_t^{+\infty} |e^{-\mu s} (q_k(s) - q(s))|^2 ds \rightarrow 0 \quad (k \rightarrow +\infty).$$

Now for  $t, T$  fixed,  $t \leq T \leq k$ , we have from (3.7),

$$p_k(T) - p_k(t) = - \int_t^T f(y(s), p_k(s)) ds + \int_t^T q_k(s) dB(s).$$

Let  $k \rightarrow +\infty$ , we obtain that

$$p(T) - p(t) = - \int_t^T f(y(s), p(s)) ds + \int_t^T q(s) dB(s).$$

That is

$$dp(t) = -f(y(t), p(t)) dt + q(t) dB(t).$$

But we know that

$$E^x |e^{-\mu t} p_k(t)|^2 \leq \frac{M^2}{\mu} e^{-2\mu t},$$

so we obtain that

$$\sup_{x \in \bar{Q}} E^x |e^{-\mu t} p(t)|^2 \leq \frac{M^2}{\mu^2} e^{-2\mu t}$$

which conclude our proof of existence and the estimate (3.4).

(2) *Uniqueness.* The proof of uniqueness is much simpler. Assume that  $(p_1(\cdot), q_1(\cdot))$ ,  $(p_2(\cdot), q_2(\cdot))$  are two solutions of (3.3). Using the above method, we can obtain that

$$\begin{aligned} & E |e^{-\mu t} (p_1(t) - p_2(t))|^2 + E \int_t^{+\infty} |e^{-\mu s} (q_1(s) - q_2(s))|^2 ds \\ &= 2E^x \int_t^{+\infty} (e^{-\mu s})^2 (\mu |p_1(s) - p_2(s)|^2 \\ & \quad + \langle f(y(s), p_1(s)) - f(y(s), p_2(s)), p_1(s) - p_2(s) \rangle) ds \leq 0 \end{aligned}$$

which implies uniqueness.  $\square$

**Theorem 3.3.** Assume that  $u$  is a  $C^2(\bar{Q}; \mathbb{R}^m)$  solution of

$$\begin{aligned} \frac{1}{2} \Delta u + f(x, u) &= 0, \quad x \in Q, \\ \partial u / \partial n &= 0, \quad x \in G, \end{aligned} \tag{3.9}$$

and  $f$  satisfies assumptions (3.1), (3.2). Then the solution  $u$  of the system (3.9) has the following probabilistic interpretation:

$$u(x) = E^x p(0) = p(0) \tag{3.10}$$

where  $(p(\cdot), q(\cdot))$  is the unique solution of (3.3).

Furthermore  $u(\cdot)$  has an a priori estimate

$$\max_{x \in \bar{Q}} |u(x)| \leq M / \mu \tag{3.11}$$

where

$$M = \max_{x \in \bar{Q}} |f(x, 0)|.$$

**Proof.** (3.10) can be proved by the same method as in Theorem 2.3 and (3.11) is a simple consequence of (3.4).  $\square$

#### 4. The general case

In this section, we discuss the general case using the results in Sections 2 and 3.

Consider the following backward SDE:

$$\begin{aligned} dp(t) &= -\frac{1}{2}\varphi(y(t))L(dt) - f(y(t), p(t)) dt + q(t) dB(t), \\ (L^2) \lim_{t \rightarrow +\infty} e^{-\mu t} p(t) &= 0, \end{aligned} \quad (4.1)$$

where  $\varphi \in C^2(G; \mathbb{R}^m)$  and  $f$  satisfies (3.1) and (3.2).

We will compare (4.1) with the linear case whose solution will be denoted by  $p^*(\cdot) = p^*(\cdot, x, \varphi) = p^*(\cdot, x)$  and  $q^*(\cdot)$ ,

$$\begin{aligned} dp^*(t) &= -\frac{1}{2}\varphi(y(t))L(dt) + \mu p^*(t) dt + q^*(t) dB(t), \\ (L^2) \lim_{t \rightarrow +\infty} e^{-\mu t} p^*(t) &= 0. \end{aligned} \quad (4.2)$$

From Section 2, we know that

$$p^*(0) = u^*(x) = u^*(x, \varphi) \quad (4.3)$$

where  $u^*(\cdot) = u^*(\cdot, \varphi)$  is the  $C^2(\bar{Q}; \mathbb{R}^m)$  solution of

$$\begin{aligned} \frac{1}{2}\Delta u^* - \mu u^* &= 0, \quad x \in Q, \\ \partial u^*/\partial n &= \varphi, \quad x \in G. \end{aligned} \quad (4.4)$$

**Lemma 4.1.** *We have a uniform estimate:*

$$\sup_{\substack{(t,x) \in \mathbb{R}^+ \times \bar{Q} \\ \omega \in \Omega}} |p^*(t, x)| \leq \max_{x \in \bar{Q}} |u^*(x, \varphi)|. \quad (4.5)$$

**Proof.** From the proof of Theorem 2.3, we know that

$$p^*(t) = u^*(y(t)) = u^*(y(t), \varphi)$$

which implies (4.5).  $\square$

**Theorem 4.2.** *Suppose that  $\varphi \in C^2(G; \mathbb{R}^m)$  and  $f$  satisfies (3.1), (3.2). Then there exists a unique solution  $(p(\cdot), q(\cdot))$  of (4.1).*

*Furthermore, we have an estimate:*

$$\sup_{x \in \bar{Q}} [E^x |p(t)|^2]^{1/2} \leq \frac{M}{\mu} + \left( \frac{c}{\mu} + 2 \right) \max_{x \in \bar{Q}} |u^*(x, \varphi)| \quad (4.6)$$

where

$$M = \max_{x \in \bar{Q}} |f(x, 0)|.$$

**Proof.** We set

$$\tilde{f}(t, x, v) = f(x, v + p^*(t, x)) + \mu p^*(t, x). \quad (4.7)$$

Then

$$|\tilde{f}(t, x, v_1) - \tilde{f}(t, x, v_2)| \leq c|v_1 - v_2|, \quad (4.8)$$

$$\langle \tilde{f}(t, x, v_1) - \tilde{f}(t, x, v_2), v_1 - v_2 \rangle \leq -\mu|v_1 - v_2|^2. \quad (4.9)$$

Using exactly the same way as in Theorem (3.1), we can deduce that there exists a unique solution  $(\tilde{p}(\cdot), \tilde{q}(\cdot))$  of

$$\begin{aligned} d\tilde{p}(t) &= -\tilde{f}(t, y(t), \tilde{p}(t)) dt + \tilde{q}(t) dB(t), \\ (L^2) \lim_{t \rightarrow +\infty} e^{-\mu t} \tilde{p}(t) &= 0, \end{aligned} \quad (4.10)$$

and

$$(E^x |\tilde{p}(t)|^2)^{1/2} \leq \frac{1}{\mu} \sup_{\substack{(t,x) \in \mathbb{R}^+ \times \bar{Q} \\ \omega \in \Omega}} |\tilde{f}(t, x, 0)|, \quad (4.11)$$

from which we obtain that

$$\begin{aligned} (E^x |\tilde{p}(t)|^2)^{1/2} &\leq \frac{1}{\mu} \sup_{\substack{(t,x) \in \mathbb{R}^+ \times \bar{Q} \\ \omega \in \Omega}} [|f(x, p^*(t, x))| + \mu |p^*(t, x)|] \\ &\leq \frac{1}{\mu} \sup_{\substack{(t,x) \in \mathbb{R}^+ \times \bar{Q} \\ \omega \in \Omega}} [|f(x, 0)| + (c + \mu) |p^*(t, x)|] \\ &\leq \frac{M}{\mu} + \left( \frac{c}{\mu} + 1 \right) \max_{x \in \bar{Q}} |u^*(x, \varphi)|. \end{aligned} \quad (4.12)$$

Now set

$$p(t) = \tilde{p}(t) + p^*(t), \quad q(t) = \tilde{q}(t) + q^*(t).$$

It is easy to verify that  $(p(\cdot), q(\cdot))$  is what we want and

$$\sup_{x \in \bar{Q}} [E^x |p(t)|^2]^{1/2} \leq \frac{M}{\mu} + \left( \frac{c}{\mu} + 2 \right) \max_{x \in \bar{Q}} |u^*(x, \varphi)|.$$

Now the existence result is established. The uniqueness can be proved in exactly the same way as in Theorem 3.1.  $\square$

Now the following theorem is evident.

**Theorem 4.3.** Assume that  $u$  is a  $C^2(\bar{Q}; \mathbb{R}^m)$  solution of

$$\begin{aligned} \frac{1}{2} \Delta u + f(x, u) &= 0, \quad x \in Q, \\ \partial u / \partial n &= \varphi, \quad x \in G, \end{aligned} \quad (4.13)$$

and  $\varphi \in C^2(G; \mathbb{R}^m)$ ,  $f$  satisfies (3.1), (3.2), then the  $C^2(\bar{Q}; \mathbb{R}^m)$  solution  $u$  of the system (4.13) has the following probabilistic interpretation:

$$u(x) = E^x p(0) = p(0), \quad (4.14)$$

where  $(p(\cdot), q(\cdot))$  is a unique solution of (4.1). Furthermore,  $u(\cdot)$  has an a priori estimate:

$$\max_{x \in \bar{Q}} |u(x)| \leq \frac{M}{\mu} + \left( \frac{c}{\mu} + 2 \right) \max_{x \in \bar{Q}} |u^*(x, \varphi)|. \quad \square \quad (4.15)$$

## 5. Existence and uniqueness for Neumann problem

In Theorem 4.3, we assumed that (4.13) had a  $C^2(\bar{Q}; \mathbb{R}^m)$  solution. In this section, we establish an existence and uniqueness result for  $C^2(\bar{Q}; \mathbb{R}^m)$  solution of (4.13) by use of a Leray-Schaude type theorem in Ural'ceva (1962) and our a priori estimate (4.15).

**Lemma 5.1.** Assume  $\varphi \in C^2(G; \mathbb{R}^m)$ ,  $f \in C^3(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^m)$ . Let it be possible to give uniformly with respect to  $\tau \in [0, 1]$  an a priori estimate of  $\max_{x \in \bar{Q}} |u(x, \tau)|$  of  $C^2(\bar{Q}; \mathbb{R}^m)$  solutions  $u(x, \tau)$  of the Neumann boundary value problems

$$\begin{aligned} \tau(\tfrac{1}{2}\Delta u + f(x, u)) + (1-\tau)(\tfrac{1}{2}\Delta u - \mu u) &= 0, \quad x \in \bar{Q}, \\ \tau\left(\frac{\partial u}{\partial n} - \varphi\right) + (1-\tau)\frac{\partial u}{\partial n} &= 0, \quad x \in G. \end{aligned} \quad (5.1)$$

Then a  $C^2(\bar{Q}; \mathbb{R}^m)$  solution to problem (4.13) exists.

**Proof.** See Ural'ceva (1962) or Ladyženskaja and Ural'ceva (1968).  $\square$

**Remark 5.2.** For simplicity of the exposition, we have not introduced the space  $C^{2,\alpha}(\bar{Q}; \mathbb{R}^m)$ , etc. and assumed some stronger smoothness assumptions here. The reader is referred to the references above for the corresponding suitable conditions for Lemma 5.1 to hold in detail.

Now we can state our main result in this section:

**Theorem 5.3.** Assume  $\varphi \in C^2(G; \mathbb{R}^m)$ ,  $f \in C^3(\mathbb{R}^n \times \mathbb{R}^m; \mathbb{R}^m)$  satisfies (3.1), (3.2). Then a  $C^2(\bar{Q}; \mathbb{R}^m)$  solution to the problem (4.13) exists uniquely and is given by

$$u(x) = E^x p(0) = p(0) \quad (5.2)$$

where  $(p(\cdot), q(\cdot))$  is the unique solution of (4.1).

**Proof.** The uniqueness and (5.2) is a direct consequence of Theorem 4.3. Now we consider (5.1) which is the same as

$$\begin{aligned} \frac{1}{2}\Delta u + \tilde{f}(x, u, \tau) &= 0, & x \in Q, \\ \partial u / \partial n &= \tau\varphi, & x \in G, \end{aligned} \quad (5.3)$$

where

$$\tilde{f}(x, v, \tau) = \tau f(x, v) - (1 - \tau)\mu v, \quad |\tilde{f}(x, 0, \tau)| = \tau |f(x, 0)|.$$

We have also

$$\tilde{f}(x, v_1, \tau) - \tilde{f}(x, v_2, \tau) = \tau(f(x, v_1) - f(x, v_2)) - (1 - \tau)\mu(v_1 - v_2).$$

Then

$$\begin{aligned} |\tilde{f}(x, v_1, \tau) - \tilde{f}(x, v_2, \tau)| &\leq \tau |f(x, v_1) - f(x, v_2)| + (1 - \tau)\mu |v_1 - v_2| \\ &\leq \tau c |v_1 - v_2| + (1 - \tau)\mu |v_1 - v_2| \\ &= [\tau c + (1 - \tau)\mu] |v_1 - v_2|, \\ \langle \tilde{f}(t, v_1, \tau) - \tilde{f}(t, v_2, \tau), v_1 - v_2 \rangle &\leq -\tau\mu |v_1 - v_2|^2 - (1 - \tau)\mu |v_1 - v_2|^2 \\ &= -\mu |v_1 - v_2|^2. \end{aligned}$$

Suppose that  $u(x, \tau)$  is a  $C^2(\bar{Q}; \mathbb{R}^m)$  solution of (5.1). Then from (4.15), we would have

$$\begin{aligned} \max_{x \in \bar{Q}} |u(x, \tau)| &\leq \tau \frac{M}{\mu} + \left( \frac{\tau c + (1 - \tau)\mu}{\mu} + 2 \right) \tau \max_{x \in \bar{Q}} |u^*(x, \varphi)| \\ &\leq \frac{M}{\mu} + \left( \frac{c}{\mu} + 3 \right) \max_{x \in \bar{Q}} |u^*(x, \varphi)|. \end{aligned}$$

Obviously this a priori estimate is uniform with respect to  $\tau$ . This concludes our proof.  $\square$

**Remark 5.4.** It is possible (but we shall not exploit this fact here) to extend our results to more general systems of PDEs under more general boundary value conditions.

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